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On the Nonlinear Examples of Sequence Spaces Induced by *Lp*-Functions

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*Lp*関数によって与えられた非線形数列集合の例について

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Abstract. In this paper, we discuss the linearity and the nonlinearity of a sequence space $\Lambda_p(f)$ induced by a L_p-function f. In particular, we give examples of L_p-functions such that $\Lambda_p(f)$ are not linear.

Introduction and preliminaries

Let $f(\neq 0)$ be an L_p-function defined on the real line $\mathbb{R}$ and assume $1 \leq p < +\infty$. For a sequence of real numbers $\boldsymbol{a} = (a_n) \in \mathbb{R}^\infty$, define

$$\Psi_p(\boldsymbol{a}; f) := \left(\sum_k \int_{\mathbb{R}} |f(x - a_k) - f(x)|^p dx \right)^{1/p}$$

and

$$\Lambda_p(f) := \{\boldsymbol{a} \in \mathbb{R}^\infty : \Psi_p(\boldsymbol{a}; f) < +\infty\}.$$

The following results are known (cf.[2]):

- For every $\boldsymbol{a} = (a_n) \in \mathbb{R}^\infty$,
  $\Psi_p(|\boldsymbol{a}|; f) = \Psi_p(\boldsymbol{a}; f)$, where $|\boldsymbol{a}| = (|a_n|)$;
- $\Psi_p(\boldsymbol{a} - \boldsymbol{b}; f) \leq \Psi_p(\boldsymbol{a}; f) + \Psi_p(\boldsymbol{b}; f)$ for every $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^\infty$, i.e, the sets $\Lambda_p(f)$ are additive subgroups of $\mathbb{R}^\infty$.

Let $W^{1,p}(\mathbb{R})$ be a Sobolev space, i.e, $f \in W^{1,p}(\mathbb{R})$ if and only if $f \in L_p(\mathbb{R})$ and the derivative Df of f in the sense of distribution belongs to $L_p(\mathbb{R})$. In particular, if $f \in L^1(\mathbb{R})$ and Df is a Radon measure of bounded variation on $\mathbb{R}$, f called a function of bounded variation. The class of all such functions will be denoted by $BV(\mathbb{R})$. Thus, $f \in BV(\mathbb{R})$ if and only if there is a Radon measure μ defined in $\mathbb{R}$ such that $|\mu|(\mathbb{R}) < +\infty$ and

$$\int_{\mathbb{R}} f\varphi' dx = -\int \varphi d\mu, \ \varphi \in C_0^\infty(\mathbb{R}),$$

where, $|Df|(\mathbb{R}) = |\mu|(\mathbb{R})$ means the total variation of μ.

It is obvious that a function f on $\mathbb{R}$ is absolutely continuous and the derivative f' is in $L_1(\mathbb{R})$, then f is of bounded variation, i.e. $W^{1,1}(\mathbb{R}) \subset BV(\mathbb{R})$ (see [5]).

In [2], A. Honda, Y. Okazaki and H. Sato provided the following results:

(i) ([2, Theorem 1, Theorem 2]) If $1 \leq p < +\infty$ and $f(\neq 0) \in L_p(\mathbb{R})$, then $\Lambda_p(f) \subset \ell_p$. In particular, $f \in W^{1,p}(\mathbb{R})$ implies $\ell_p = \Lambda_p(f)$.

(ii) ([2, Corollary 4]) If $1 < p < +\infty$ and $f(\neq 0) \in L_p(\mathbb{R})$, then $\ell_p = \Lambda_p(f)$ if and only if $f \in W^{1,p}(\mathbb{R})$.

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On the other hand, the authors ([6]) showed that necessary and sufficient conditions for the linearity of $\Lambda_p(f)$, and that $\ell_1 = \Lambda_1(f)$ if and only if $f \in BV(\mathbb{R})$.

In this paper, we discuss the linearity and the nonlinearity of a sequence space $\Lambda_p(f)$ induced by a L_p-function f. In particular, we give two examples of L_p-functions such that $\Lambda_p(f)$ are not linear.

1. The linearity of $\Lambda_p(f)$

In [6], We gave necessary and sufficient conditions for the linearity of $\Lambda_p(f)$, and an example such that $\Lambda_p(f)$ is linear.

Theorem 1.1. ([6, Theorem 2.1]) *Let $1 \le p < +\infty$ and $f(\neq 0) \in L_p(\mathbb{R})$. Then the following are equivalent:*

(i) *$\Lambda_p(f)$ is a linear subspace of $\mathbb{R}^\infty$;*

(ii) *For any $0 \le k \le 1$, there exists a constant $C(k) > 0$ such that*

$$\begin{aligned}&\int_{\mathbb{R}} |f(x-ka) - f(x)|^p dx \\ \le \ & C(k) \int_{\mathbb{R}} |f(x-a) - f(x)|^p dx, \ \forall a > 0;\end{aligned}$$

(iii) *There exits a constant $C > 0$ such that*

$$\begin{aligned}&\int_{\mathbb{R}} |f(x-ka) - f(x)|^p dx \\ \le \ & C \int_{\mathbb{R}} |f(x-a) - f(x)|^p dx, \ 0 \le \forall k \le 1, \ \forall a > 0.\end{aligned}$$

Theorem 1.2. ([3] and [6, Theorem 2.2]) *Let $f \in L^p(\mathbb{R})$, $1 \le p < \infty$. If there exists a countable partition $(a_i)_{-\infty}^{\infty}$ on $\mathbb{R}$ satisfying the following conditions:*

(1) *$a_i < a_{i+1}$ and $\lim_{i \to \pm\infty} a_i = \pm\infty$;*

(2) *$\inf_i (a_{i+1} - a_i) > 0$;*

(3) *f is monotone on (a_i, a_{i+1}).*

Then $\Lambda_p(f)$ is linear.

2. The nonlinearity of $\Lambda_p(f)$

In this section, we give two examples of L_p-functions such that $\Lambda_p(f)$ are not linear. We begin with a following theorem.

Theorem 2.1. *Let $f_0 \in C_0(\mathbb{R})(\neq 0)$ with $\operatorname{supp} f_0 \subset [0, \pi]$. For m and $n \in \mathbb{N}$, we define $f_{m,n} \in C(\mathbb{R})$ by*

$$f_{m,n}(x) = 1 + \frac{1}{m}\sin(nx).$$

Then there exist sequences $\{m_i\}$ and $\{n_i\}$ satisfying the following conditions (i) *and* (ii)*:*

(i) $f(x) = \lim_{j \to \infty} f_0(x) \prod_{i=1}^{j} f_{m_i,n_i}(x)$ *(uniformly on $\mathbb{R}$).*

(ii) $\lim_{i \to \infty} \dfrac{\|f(\cdot - \pi/n_i) - f(\cdot)\|_p}{\|f(\cdot - 2\pi/n_i) - f(\cdot)\|_p} = \infty.$

Proof. Let $0 < \alpha < 1$ and $\beta > 1$ and take a sequence (m_i) of $\mathbb{N}$ so that

$$0 < \alpha \le \prod_{i=1}^{\infty} \left(1 - \frac{1}{m_i}\right) \le 1 \le \prod_{i=1}^{\infty} \left(1 + \frac{1}{m_i}\right) \le \beta. \tag{2.1}$$

On the other hand, determine inductively a sequence (n_j) of $\mathbb{N}$ satisfying the following conditions:

$$f_j(x) = f_0(x) \prod_{i=1}^{j} \left(1 + \frac{1}{m_i} \sin(n_i x)\right), \tag{2.2}$$

$$\left\| f_j\left(\cdot - \frac{\pi}{n_j}\right) - f_j(\cdot) \right\|_p \ge (j-1) \left\| f_j\left(\cdot - \frac{2\pi}{n_j}\right) - f_j(\cdot) \right\|_p, \tag{2.3}$$

(2.4) $\quad n_i$ is a multiple of $2n_{i-1}$ for every $2 \le i \le j$.

First, put $n_1 = 1$ and assume that the above three conditions (2.2), (2.3) and (2.4) hold for $n_1, n_2, \cdots, n_j$. We define $f_{j,n}$ as follows:

$$f_{j,n}(x) = f_j(x) \left(1 + \frac{1}{m_{j+1}} \sin(nx)\right) \qquad \text{for } n \in \mathbb{N},\ x \in \mathbb{R} \tag{2.5}$$

Then we have

$$\begin{aligned} & f_{j,n}\left(x - \frac{\pi}{n}\right) - f_{j,n}(x) \\ = \ & f_j\left(x - \frac{\pi}{n}\right)\left(1 + \frac{1}{m_{j+1}} \sin(nx - \pi)\right) - f_j(x)\left(1 + \frac{1}{m_{j+1}} \sin(nx)\right) \\ = \ & f_j\left(x - \frac{\pi}{n}\right)\left(1 - \frac{1}{m_{j+1}} \sin(nx)\right) - f_j(x)\left(1 + \frac{1}{m_{j+1}} \sin(nx)\right) \\ = \ & \left(f_j\left(x - \frac{\pi}{n}\right) - f_j(x)\right)\left(1 - \frac{1}{m_{j+1}} \sin(nx)\right) - \frac{2}{m_{j+1}} f_j(x) \sin(nx). \end{aligned}$$

Since $f_j (\neq 0) \in C_0(\mathbb{R})$ and

$$\lim_{n \to \infty} \left\| f_j\left(\cdot - \frac{\pi}{n}\right) - f_j(\cdot) \right\|_p = 0,$$

we see that

$$\begin{aligned} \liminf_{n \to \infty} \left\| f_{j,n}\left(\cdot - \frac{\pi}{n}\right) - f_{j,n}(\cdot) \right\|_p &= \frac{2}{m_{j+1}} \liminf_{n \to \infty} \left(\int_{\mathbb{R}} |f_j(x) \sin(nx)|^p \, dx \right)^{1/p} \\ &= \frac{2}{m_{j+1}} \|f_j\|_p \left(\frac{1}{\pi} \int_0^{\pi} |\sin x|^p dx \right)^{1/p} > 0. \end{aligned} \tag{2.6}$$

On the other hand,

$$\begin{aligned} & f_{j,n}\left(x - \frac{2\pi}{n}\right) - f_{j,n}(x) \\ = \ & f_j\left(x - \frac{2\pi}{n}\right)\left(1 + \frac{1}{m_{j+1}} \sin(nx - 2\pi)\right) - f_j(x)\left(1 + \frac{1}{m_{j+1}} \sin(nx)\right) \\ = \ & \left\{ f_j\left(x - \frac{2\pi}{n}\right) - f_j(x) \right\}\left(1 + \frac{1}{m_{j+1}} \sin(nx)\right). \end{aligned}$$

Since

$$\lim_{n \to \infty} \left\| f_j\left(\cdot - \frac{2\pi}{n}\right) - f_j(\cdot) \right\|_p = 0,$$

we have

$$(2.7)\qquad \lim_{n\to\infty}\left\|f_{j,n}\left(\cdot-\frac{2\pi}{n}\right)-f_{j,n}(\cdot)\right\|_p=0.$$

From (2.6) and (2.7), we see that

$$(2.8)\quad \left\|f_{j,n_{j+1}}\left(\cdot-\frac{\pi}{n_{j+1}}\right)-f_{j,n_{j+1}}(\cdot)\right\|_p\geq j\left\|f_{j,n_{j+1}}\left(\cdot-\frac{2\pi}{n_{j+1}}\right)-f_{j,n_{j+1}}(\cdot)\right\|_p$$

for a sufficiently large number n_{j+1} with a multiple of $2n_j$.

From the definition of $f_{j,n_{j+1}}$,

$$\begin{aligned} f_{j,n_{j+1}}(x) &= f_j(x)\left(1+\frac{1}{m_{j+1}}\sin(n_{j+1}\,x)\right)\\ &= f_0(x)\prod_{i=1}^{j}\left(1+\frac{1}{m_i}\sin(n_i\,x)\right)\left(1+\frac{1}{j+1}\sin(n_{j+1}\,x)\right)\\ &= f_{j+1}(x). \end{aligned}$$

Thus we see that (2.8) implies

$$\left\|f_{j+1}\left(\cdot-\frac{\pi}{n_{j+1}}\right)-f_{j+1}(\cdot)\right\|_p\geq j\left\|f_{j+1}\left(\cdot-\frac{2\pi}{n_{j+1}}\right)-f_{j+1}(\cdot)\right\|_p.$$

Hence we see that (2.2), (2.3) and (2.4) hold for $j+1$. Define $f(x)$ on $\mathbb{R}$ by

$$(2.9)\qquad f(x)=\lim_{j\to\infty}f_j(x)=\lim_{j\to\infty}f_0(x)\prod_{i=1}^{j}\left(1+\frac{1}{m_i}\sin(n_i x)\right),$$

where the convergence is uniform on $\mathbb{R}$ by (2.1). Then it is obvious from (2.1) that $f\in C_0(\mathbb{R})$ and (i) holds. For $j\in\mathbb{N}$ and $x\in\mathbb{R}$, we have

$$\begin{aligned} & f\left(x-\pi/n_j\right)-f(x)\\ =\ & f_j\left(x-\pi/n_j\right)\prod_{i=j+1}^{\infty}\left(1+\frac{1}{m_i}\sin\left(n_i\left(x-\pi/n_j\right)\right)\right)\\ & -f_j(x)\prod_{i=j+1}^{\infty}\left(1+\frac{1}{m_i}\sin\left(n_i x\right)\right)\\ =\ & \left(f_j\left(x-\pi/n_j\right)-f_j(x)\right)\prod_{i=j+1}^{\infty}\left(1+\frac{1}{m_i}\sin\left(n_i x\right)\right). \end{aligned}$$

And hence from (2.1) we have

$$\begin{aligned} \left|f\left(x-\pi/n_j\right)-f(x)\right| &= \left|\left(f_j\left(x-\pi/n_j\right)-f_j(x)\right)\right|\prod_{i=j+1}^{\infty}\left|1+\frac{1}{m_i}\sin\left(n_i x\right)\right|\\ &\geq \alpha\left|f_j\left(x-\pi/n_j\right)-f_j(x)\right|, \end{aligned}$$

and so

$$(2.10)\qquad \left\|f\left(\cdot-\pi/n_j\right)-f(\cdot)\right\|_p\geq\alpha\left\|f_j\left(\cdot-\pi/n_j\right)-f_j(\cdot)\right\|_p\quad\text{for } j\in\mathbb{N}.$$

In the same way, we have

$$\begin{aligned} & f\left(x-2\pi/n_j\right)-f(x) \\ = \; & f_j\left(x-2\pi/n_j\right)\prod_{i=j+1}^{\infty}\left(1+\frac{1}{m_i}\sin\left(n_i\left(x-2\pi/n_j\right)\right)\right) \\ & -f_j(x)\prod_{i=j+1}^{\infty}\left(1+\frac{1}{m_i}\sin\left(n_i x\right)\right) \\ = \; & \left(f_j\left(x-2\pi/n_j\right)-f_j(x)\right)\prod_{i=j+1}^{\infty}\left(1+\frac{1}{m_i}\sin\left(n_i x\right)\right). \end{aligned}$$

And hence from (2.1) we have

$$\begin{aligned} \left|f\left(x-2\pi/n_j\right)-f(x)\right| &= \left|\left(f_j\left(x-2\pi/n_j\right)-f_j(x)\right)\right|\prod_{i=j+1}^{\infty}\left|1+\frac{1}{m_i}\sin\left(n_i x\right)\right| \\ &\le \left|\left(f_j\left(x-2\pi/n_j\right)-f_j(x)\right)\right|\prod_{i=j+1}^{\infty}\left(1+\frac{1}{m_i}\right) \\ &\le \beta\left|f_j\left(x-2\pi/n_j\right)-f_j(x)\right|, \end{aligned}$$

and so

$$\left\|f_j\left(\cdot-2\pi/n_j\right)-f_j(\cdot)\right\|_p \ge (1/\beta)\left\|f\left(\cdot-2\pi/n_j\right)-f(\cdot)\right\|_p \quad \text{for all } j\in\mathbb{N}. \tag{2.11}$$

Combining (2.3), (2.10) and (2.11), we have

$$\left\|f\left(\cdot-\pi/n_j\right)-f(\cdot)\right\|_p \ge (j-1)(\alpha/\beta)\left\|f\left(\cdot-2\pi/n_j\right)-f(\cdot)\right\|_p \tag{2.12}$$

for all $j\in\mathbb{N}$. Thus we see that (ii) holds. □

Remark 2.2. *We see easily from* (2.12) *that there does not exist a constant C such that*

$$\|f(\cdot-a/2)-f(\cdot)\|_p^p \le C\|f(\cdot-a)-f(\cdot)\|_p^p$$

holds for every $a>0$. Thus we see from condition (iii) *of Theorem* 1.1 *that $\Lambda_p(f)$ is not a linear subspace in $\mathbb{R}^\infty$.*

Next we give an example of a more smooth function f such that $\Lambda_p(f)$ is not linear. We begin with a lemma.

Lemma 2.3. *Let $1\le p<\infty$ and $-\infty<a\le\infty$. Let $\psi\in L_p((-\infty,a))\cap C^1((-\infty,a))$ and $\psi'\in L_p((-\infty,a))$. Then for any $c>0$, we have*

$$\|\psi(\cdot-c)-\psi(\cdot)\|_{L_p((-\infty,a))} \le c\|\psi'\|_{L_p((-\infty,a))}.$$

Proof. For $x\in(-\infty,a)$, we have

$$\begin{aligned} |\psi(x)-\psi(x-c)| = \left|\int_{x-c}^{x}\psi'(t)\,dt\right| &\le \int_{x-c}^{x}|\psi'(t)|\,dt \\ &\le \left(\int_{x-c}^{x}|\psi'(t)|^p\,dt\right)^{1/p}\left(\int_{x-c}^{x}1^q\,dt\right)^{1/q} \\ &= c^{1/q}\left(\int_{x-c}^{x}|\psi'(t)|^p\,dt\right)^{1/p}, \end{aligned}$$

where $1/p + 1/q = 1$. Hence

$$\begin{aligned}
\|\psi(\cdot) - \psi(\cdot - c)\|^p_{L_p((-\infty,a))} &= \int_{-\infty}^{a} |\psi(x) - \psi(x-c)|^p \, dt \\
&\leq c^{p/q} \int_{-\infty}^{a} \int_{x-c}^{x} |\psi'(t)|^p \, dt \, dx \\
&\leq c^{p/q} \int_{-\infty}^{a} \int_{t}^{t+c} |\psi'(t)|^p \, dx \, dt \\
&\leq c^{p/q+1} \int_{-\infty}^{a} |\psi'(t)|^p \, dt.
\end{aligned}$$

Thus we have

$$\|\psi(\cdot - c) - \psi(\cdot)\|_{L_p((-\infty,a))} \leq c\|\psi'\|_{L_p((-\infty,a))}.$$

□

Theorem 2.4. *Let $1 \leq p < \infty$. Then there exist a function $f \in L_p(\mathbb{R})$ and a sequence (n_j) of $\mathbb{N}$ such that:*
(i) *$f \in C^\infty(\mathbb{R}) \cap L_p(\mathbb{R})$ and $f(x) > 0$ ($x \in \mathbb{R}$);*
(ii) *the number of x satisfying $f'(x) = 0$ on every bounded subinterval I of $\mathbb{R}$ is finite;*
(iii) $\displaystyle\lim_{k\to\infty} \frac{\|f(\cdot - 1/n_k) - f(\cdot)\|_p}{\|f(\cdot - 2/n_k) - f(\cdot)\|_p} = \infty.$

Proof. We can construct f as follows. Let

$$\rho(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & (-1 < x < 1) \\ 0 & |x| \geq 1. \end{cases}$$

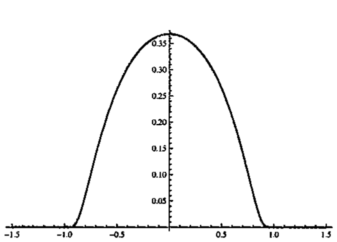

Fig.1 $y = \rho(x)$

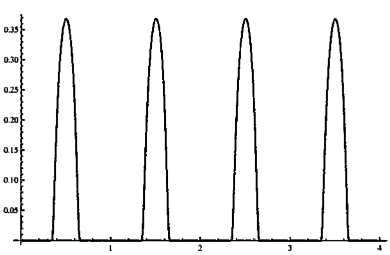

Fig.2 $y = \rho_n(x)$

Then $\rho \in C_0^\infty(\mathbb{R})$ and $\operatorname{supp} \rho = [-1.1]$. Moreover, for all $n \in \mathbb{N}$, let $\rho_n(x) = \rho(6(x - n + 1/2))$, then we have $\operatorname{supp} \rho_n = [n - 2/3, n - 1/3]$ and $0 \leq \rho_n(x) \leq 1/e$.

Next, choose a sequence (n_k) so that n_k is a multiple of n_{k-1} for each $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} e^{-k^2} \frac{n_k}{n_{k-1}} = \infty \tag{2.13}$$

holds (for example, $n_k = (k!)!$).

Put

$$\begin{aligned}\varphi(x) &:= e^{-x^2}\\ g(x) &:= \varphi(x)\sum_{k=1}^{\infty}\rho_k(x)\\ h(x) &:= \varphi(x)\sum_{k=1}^{\infty}\rho_k(x)\sin(n_k\pi x)\\ f(x) &:= \varphi(x)+h(x).\end{aligned}$$

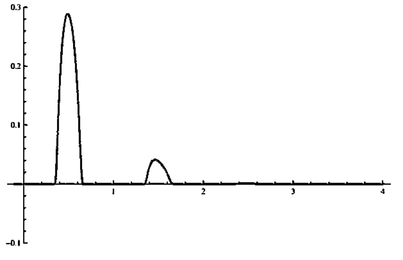

Fig.3 $y = g(x)$

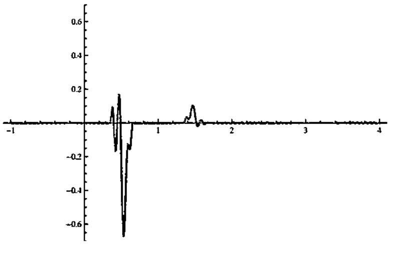

Fig.4 $y = h(x)$

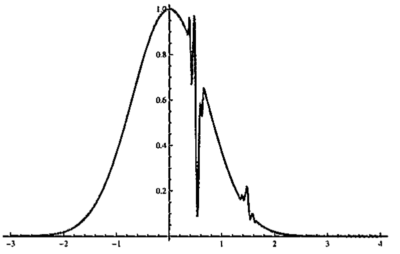

Fig.5 $y = f(x) = \varphi(x) + h(x)$

We see easily from $0 \le \rho_k(x) \le 1/e$ and the definition of f that $f(x) > 0$ on $\mathbb{R}$ and $f \in C^\infty(\mathbb{R}) \cap L_p(\mathbb{R})$ and so (i) hlods.

To show that (ii), let $k \in \mathbb{N}$. We should note that $f(x) = \varphi(x)$ on $(k-1, k-2/3] \cup [k-1/3, k]$. We see from the definition that $f'(x) = \varphi'(x) \ne 0$ on $(k-1, k-2/3]$, $[k-1/3, k)$.

Next, we show that $\{x : f'(x) = 0, k-2/3 < x < k-1/3\}$ is a finite set. In fact, suppose that the set $\{x : f'(x) = 0, k-2/3 < x < k-1/3\}$ is infinite. Since $f'(k-2/3) = \varphi'(k-2/3) \ne 0$ and $f'(k-1/3) = \varphi'(k-1/3) \ne 0$, we see that an accumulation point is in $(k-2/3, k-1/3)$. Put

$$f(z) = e^{-z^2}(1 + e^{-1/\left(1-36(z-k+1/2)^2\right)}\sin(n_k\pi z))$$

on $z \in \mathbb{C}$. Then we see that $f(z)$ is regular on $\mathbb{C} \setminus \{k-2/3, k-1/3\}$ and We see from the identity theorem that $f'(z) = 0$ on $\mathbb{C} \setminus \{k-2/3, k-1/3\}$. Hence

$f'(k-2/3) = \lim_{\varepsilon\to+0} f'(k-2/3+\varepsilon) = 0$, which contradicts the hypothesis. Thus, we see that the set $\{x : f'(x) = 0, k-2/3 < x < k-1/3\}$ for every $k \in \mathbb{N}$ is finite, and so the set $\{x : f'(x) = 0, k-1 \le x < k\}$ for every $k \in \mathbb{N}$ is also finite. Moreover, we see that the number of x satisfying $f'(x) = 0$ on every bounded subinterval of $\mathbb{R}$ is finite.

To show (iii), We note that $\max_{k-1\le x\le k} |h'(x)| \le Mn_k$ for some $M > 0$, and so $\|h'\|_{L_p((k-1,k))} \le Mn_k$ for $k \in \mathbb{N}$.

For $k \ge 3$, we have

$$\begin{aligned} \|h'\|_{L_p((-\infty,k-1))} &\le \|h'\|_{L_p((-\infty,0))} + \sum_{i=1}^{k-2} \|h'\|_{L_p((i-1,i))} + \|h'\|_{L_p((k-2,k-1))} \\ &\le 0 + Mn_{k-2}(k-2) + Mn_{k-1}, \end{aligned}$$

and so

$$\begin{aligned} \frac{e^{k^2}}{n_k}\|h'\|_{L_p((-\infty,k-1))} &\le M\frac{e^{k^2}}{n_k}(n_{k-2}(k-2) + n_{k-1}) \\ &= Me^{k^2}\frac{n_{k-1}}{n_k}\left(e^{(k-1)^2}\frac{n_{k-2}}{n_{k-1}}e^{-(k-1)^2}(k-2) + 1\right). \end{aligned}$$

Thus we see from (2.13) that $\lim_{k\to\infty} \frac{e^{k^2}}{n_k}\|h'\|_{L_p((-\infty,k-1))} = 0$. Let $k \in \mathbb{N}$. Put $a = k-1$ and $c = 2/n_k$ for h in place of ψ in Lemma 2.3, then we have

$$\left\|h\left(\cdot - \frac{2}{n_k}\right) - h(\cdot)\right\|_{L_p((-\infty,k-1))} \le \frac{2}{n_k}\|h'\|_{L_p((-\infty,k-1))} \quad \text{for } k \in \mathbb{N}$$

In the same manner, apply Lemma 2.3 as $a = \infty$ and $c = 2/n_k$ for φ and g, then we have

$$\left\|\varphi\left(\cdot - \frac{2}{n_k}\right) - \varphi(\cdot)\right\|_p \le \frac{2}{n_k}\|\varphi'\|_p \quad \text{for } k \in \mathbb{N} \tag{2.14}$$

$$\left\|g\left(\cdot - \frac{2}{n_k}\right) - g(\cdot)\right\|_p \le \frac{2}{n_k}\|g'\|_p \quad \text{for } k \in \mathbb{N} \tag{2.15}$$

Hence

$$\begin{aligned} &\limsup_{k\to\infty} e^{k^2}\left\|f\left(\cdot - \frac{2}{n_k}\right) - f(\cdot)\right\|_p \\ \le\ &\limsup_{k\to\infty} e^{k^2}\left\|\varphi\left(\cdot - \frac{2}{n_k}\right) - \varphi(\cdot)\right\|_p \\ &+\limsup_{k\to\infty} e^{k^2}\left\|h\left(\cdot - \frac{2}{n_k}\right) - h(\cdot)\right\|_{L_p((-\infty,k-1))} \\ &+\limsup_{k\to\infty} e^{k^2}\left\|h\left(\cdot - \frac{2}{n_k}\right) - h(\cdot)\right\|_{L_p((k-1,\infty))} \\ =\ &0 + 0 + \limsup_{k\to\infty} e^{k^2}\left\|h\left(\cdot - \frac{2}{n_k}\right) - h(\cdot)\right\|_{L_p((k-1,\infty))} \\ =\ &\limsup_{k\to\infty} e^{k^2}\left\|h\left(\cdot - \frac{2}{n_k}\right) - h(\cdot)\right\|_{L_p((k-1,\infty))}. \end{aligned} \tag{2.16}$$

Put a sufficiently large $n_k \in \mathbb{N}$ so that $2/n_k < 1/3$, and let $i \in \mathbb{N}$ with $k \leq i$. Then for any $x \in [i-1, i]$, since n_i is a multiple of n_k, we have

$$h\left(x - \frac{2}{n_k}\right) = g\left(x - \frac{2}{n_k}\right) \sin\left(n_i \pi \left(x - \frac{2}{n_k}\right)\right) = g\left(x - \frac{2}{n_k}\right) \sin(n_i \pi x),$$

and so

$$\left|h\left(x - \frac{2}{n_k}\right) - h(x)\right| = \left|\left(g\left(x - \frac{2}{n_k}\right) - g(x)\right) \sin(n_i \pi x)\right| \leq \left|g\left(x - \frac{2}{n_k}\right) - g(x)\right|.$$

Thus we have that for a sufficiently large $n_k \in \mathbb{N}$,

$$\left|h\left(x - \frac{2}{n_k}\right) - h(x)\right| \leq \left|g\left(x - \frac{2}{n_k}\right) - g(x)\right| \quad \text{for all } x \geq k-1,$$

and hence

$$\begin{aligned} \left\|h\left(\cdot - \frac{2}{n_k}\right) - h(\cdot)\right\|_{L_p((k-1,\infty))} &\leq \left\|g\left(\cdot - \frac{2}{n_k}\right) - g(\cdot)\right\|_{L_p((k-1,\infty))} \\ &\leq \left\|g\left(\cdot - \frac{2}{n_k}\right) - g(\cdot)\right\|_p. \end{aligned}$$

Consequently,

$$\begin{aligned} \limsup_{k\to\infty} e^{k^2} \left\|h\left(\cdot - \frac{2}{n_k}\right) - h(\cdot)\right\|_{L_p((k-1,\infty))} &\leq \limsup_{k\to\infty} e^{k^2} \left\|g\left(\cdot - \frac{2}{n_k}\right) - g(\cdot)\right\|_p \\ &\leq \limsup_{k\to\infty} \frac{2e^{k^2}}{n_k} \|g'\|_p \\ &= 0. \quad \text{(because of (2.13) and (2.15))} \end{aligned}$$

Combining this with (2.16), we have

$$\lim_{k\to\infty} e^{k^2} \left\|f\left(\cdot - \frac{2}{n_k}\right) - f(\cdot)\right\|_p = 0. \tag{2.17}$$

Again, in the same manner, apply Lemma 2.3 as $a = \infty$ and $c = 1/n_k$ for φ and g, then we have

$$\left\|\varphi\left(\cdot - \frac{1}{n_k}\right) - \varphi(\cdot)\right\|_p \leq \frac{1}{n_k} \|\varphi'\|_p \quad \text{for all } k \in \mathbb{N}, \tag{2.18}$$

$$\left\|g\left(\cdot - \frac{1}{n_k}\right) - g(\cdot)\right\|_p \leq \frac{1}{n_k} \|g'\|_p \quad \text{for all } k \in \mathbb{N}. \tag{2.19}$$

Therefore we have

$$\begin{aligned} &\liminf_{k\to\infty} e^{k^2} \left\|f\left(\cdot - \frac{1}{n_k}\right) - f(\cdot)\right\|_p \\ \geq\ &\liminf_{k\to\infty} e^{k^2} \left(-\left\|\varphi\left(\cdot - /n_k\right) - \varphi(\cdot)\right\|_p + \left\|h\left(\cdot - /n_k\right)\right) - h(\cdot)\right\|_p\right) \\ =\ &\liminf_{k\to\infty} e^{k^2} \left\|h\left(\cdot - \frac{1}{n_k}\right) - h(\cdot)\right\|_p \\ \geq\ &\liminf_{k\to\infty} e^{k^2} \left\|h\left(\cdot - \frac{1}{n_k}\right) - h(\cdot)\right\|_{L_p((k-1,k))}. \end{aligned} \tag{2.20}$$

Put a sufficiently large $n_k \in \mathbb{N}$ so that $1/n_k < 1/3$. Then for any $x \in [k-1, k]$, we have

$$h\left(x - \frac{1}{n_k}\right) = g\left(x - \frac{1}{n_k}\right) \sin\left(n_k \pi \left(x - \frac{1}{n_k}\right)\right) = -g\left(x - \frac{1}{n_k}\right) \sin(n_k \pi x),$$

and so

$$h\left(x - \frac{1}{n_k}\right) - h(x) = -\left(g\left(x - \frac{1}{n_k}\right) + g(x)\right) \sin(n_k \pi x).$$

Hence we obtain

$$\begin{aligned}
& \left\| h(\cdot - \frac{1}{n_k}) - h(\cdot) \right\|_{L_p((k-1,k))} \\
= & \left(\int_{k-1}^{k} |(g(x) + g(x - \frac{1}{n_k})) \sin(n_k \pi x)|^p \, dx \right)^{1/p} \\
\geq & \left(\int_{k-1}^{k} |(2g(x) \sin(n_k \pi x)|^p \, dx \right)^{1/p} \\
& - \left(\int_{k-1}^{k} \left| \left(g(x - \frac{1}{n_k}) - g(x) \right) \sin(n_k \pi x) \right|^p \, dx \right)^{1/p} \\
\geq & \ 2 \left(\int_{k-1}^{k} |(g(x) \sin(n_k \pi x)|^p \, dx \right)^{1/p} - \left\| g\left(\cdot - \frac{1}{n_k}\right) - g(\cdot) \right\|_p .
\end{aligned}$$

Thus, we have

$$\begin{aligned}
(2.21) \qquad & \liminf_{k \to \infty} e^{k^2} \left\| h\left(\cdot - \frac{1}{n_k}\right) - h(\cdot) \right\|_{L_p((k-1,k))} \\
\geq & \liminf_{k \to \infty} \left\{ 2e^{k^2} \left(\int_{k-1}^{k} (g(x))^p |\sin(n_k \pi x)|^p \, dx \right)^{1/p} \right. \\
& \qquad \left. - e^{k^2} \left\| g\left(\cdot - \frac{1}{n_k}\right) - g(\cdot) \right\|_p \right\} \\
= & \liminf_{k \to \infty} 2e^{k^2} \left(\int_{k-1}^{k} (g(x))^p |\sin(n_k \pi x)|^p \, dx \right)^{1/p} \\
& \qquad \text{(because of (2.13) and (2.19))} \\
= & \liminf_{k \to \infty} 2e^{k^2} \left(\int_{k-1}^{k} (e^{-x^2} \rho_k(x))^p |\sin(n_k \pi x)|^p \, dx \right)^{1/p} \\
\geq & \liminf_{k \to \infty} 2 \left(\int_{k-1}^{k} (\rho_k(x))^p |\sin(n_k \pi x)|^p \, dx \right)^{1/p} \\
= & \liminf_{k \to \infty} 2 \left(\int_{0}^{1} (\rho_1(x))^p |\sin(n_k \pi x)|^p \, dx \right)^{1/p} \\
= & \ 2 \left(\int_{0}^{1} (\rho_1(x))^p \, dx \right)^{1/p} \left(\int_{0}^{1} |\sin \pi x|^p \, dx \right)^{1/p} > 0.
\end{aligned}$$

Consequently, it follows from (2.17) and (2.21) that

$$\lim_{k\to\infty} \frac{\|f(\cdot - 1/n_k) - f(\cdot)\|_p}{\|f(\cdot - 2/n_k) - f(\cdot)\|_p} = \lim_{k\to\infty} \frac{e^{k^2}\|f(\cdot - 1/n_k) - f(\cdot)\|_p}{e^{k^2}\|f(\cdot - 2/n_k) - f(\cdot)\|_p} = \infty, \tag{2.22}$$

which implies (iii). □

Remark 2.5. *We see easily from* (2.22) *that there does not exist a constant* C *such that*

$$\|f(\cdot - a/2) - f(\cdot)\|_p^p \leq C\|f(\cdot - a) - f(\cdot)\|_p^p$$

holds for every $a > 0$. *Thus we see from condition* (iii) *of Theorem* 1.1 *that* $\Lambda_p(f)$ *is not a linear subspace in* $\mathbb{R}^\infty$.

Moreover, we should note that Theorem 2.4 *means that condition* (2) *of Theorem* 1.2 *is essential.*

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